Week 12 - Weak Equivalences

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Contents

1 Instructions

There are five exercises. This is the last of the twelve total exercise sheets. Please submit it on Monday 27^{th}. The last lecture on Monday is optional but this sheet is not.

2 Weak Equivalences

In this exercise sheet all spaces and maps will be based. Recall that CW complexes are well-pointed when given any basepoint, and in particular a path-connected CW complex has a well-defined pointed homotopy type. For convenience we will always assume that each CW complex is pointed at a 0-cell. A connected CW complex is (pointed) homotopy equivalent to a CW complex with exactly one 0-cell and the same number of cells in all dimensions ≥ 2. For such a CW complex X there is, up to homotopy equivalence, no loss of generality in assuming all attaching maps are based. Thus the $(n+1)$-skeleton X_{n+1} is obtained as the mapping cone of a based map $\bigvee S^n \to X_n$.

We start the sheet with the following observation.

Proposition 2.1 *A map $f : X \to Y$ is a homotopy equivalence if and only if for each space K the induced map $f_* : [K, X] \to [K, Y]$ is bijective.* ∎

In general it is difficult to verify that a map is a homotopy equivalence. It would be desirable to have more easily checked criteria to decide whether a given map is a homotopy equivalence. This is the idea behind the introduction of *weak homotopy equivalences*. At the price of restricting ourselves to CW complexes we find that they give algebraic criteria to check whether a map is a homotopy equivalence.

Definition 1 *A map $\alpha : X \to Y$ of path-connected spaces X, Y is said to be a* ***weak equivalence*** *if for each CW complex K the induced function*

$$\alpha_* : [K, X] \to [K, Y] \tag{2.1}$$

is bijective. $\square$

We restrict to path-connected spaces to make the definition more useful. A definition could be formulated for non-path-connected spaces by considering path components individually.

Clearly any homotopy equivalence is a weak equivalence. There is also a partial converse.

Exercise 2.1 Show that a weak equivalence $\alpha : X \to Y$ between connected CW complexes is a homotopy equivalence. $\square$

On the other hand, not every weak equivalence is a homotopy equivalence.

Example 2.1 *The digital circle $\mathbb{S}^1$ is the finite topological space with four points which is obtained from S^1 by identifying the open northern and southern hemispheres to separate points. The digital circle is path-connected and semi-locally simply connected, so has a universal cover. This can be constructed by hand and shown to be contractible. A short computation shows that $\pi_1\mathbb{S}^1 \cong \mathbb{Z}$ and $\pi_k\mathbb{S}^1 = 0$ for $k \geq 2$. A choice of generator for $\pi_1\mathbb{S}^1$ is thus a weak equivalence $S^1 \to \mathbb{S}^1$. However there can be no non-constant map in the opposite direction since S^1 is Hausdorff and $\mathbb{S}^1$ is not.* $\square$

Weak equivalences satisfy the so called *two-of-three property*. Namely that if $f : X \to Y$ and $g : Y \to Z$ are maps between path-connected spaces, and if any two of the maps f, g, gf are weak equivalences, then so is the third. However, 'weakly equivalent to' does not define an equivalence relation in the same way that 'homotopy equivalent to' does. This is because presence of a weak equivalence $X \to Y$ does not imply the existence of a weak equivalence in the opposite direction (cf. Example 2.1).

Exercise 2.2 Assume that $\alpha : X \to Y$ is a weak equivalence between path-connected spaces. Show that for any CW complex K it induces a bijection $\alpha_* : [K, X]_0 \to [X, Y]_0$ between unpointed homotopy classes of maps. What conditions are needed to guarantee that the converse is true? $\square$

The following result is the statement which was promised above. Necessity is immediate, and the case that X, Y are finite dimensional cell complexes follows without much work. The technical details needed to cover the infinite dimensional case will be covered in the final lecture on Monday.

Theorem 2.2 *A map $\alpha : X \to Y$ between path-connected spaces X, Y is a weak equivalence if and only if for each $k \geq 1$ the induced homomorphism*

$$\alpha_* : \pi_k X \to \pi_k Y \tag{2.2}$$

is an isomorphism. ∎

Thus in completing Exercise 2.1 you have proved the following important result.

Corollary 2.3 (Whitehead) *A map $\alpha : X \to Y$ between connected CW complexes is a homotopy equivalence if and only if for each $k \geq 1$, the induced homomorphism*

$$\alpha_* : \pi_k X \to \pi_k Y \tag{2.3}$$

is an isomorphism. ∎

Corollary 2.4 *A connected CW complex X is contractible if and only if $\pi_k X = 0$ for each $k \geq 1$.* ∎

A space satisfying the conditions of the corollary, i.e. which is weakly equivalent to a point is said to be **weakly contractible**.

Example 2.2 *The infinite sphere S^∞ was defined to be the CW complex obtained as the colimit of the inclusions $\ldots \subseteq S^n \subseteq S^{n+1} \subseteq \ldots$. To fix a cell structure on S^∞ we can give S^n the CW structure with two cells in each dimension ≥ 0. When we first encountered it we showed directly that S^∞ is contractible. Here is a cuter argument:*

Any map $f : S^k \to S^\infty$ factors through some a compact subset, and hence some finite S^n. Since the inclusion $S^n \hookrightarrow S^\infty$ factors through S^{n+1} and $\pi_n S^{n+1} = 0$ we get that $\pi_k S^\infty = 0$ for each $k \geq 1$. In particular S^∞ is weakly contractible. As a CW complex it is therefore contractible. □

Although verifying that a map is a weak equivalence is an easier task than showing it to be a homotopy equivelance directly, it is still not a task to be taken lightly. A sensible way to further generalise definition 1 and make it more approachable is the following.

Definition 2 *A map $f : X \to Y$ between path-connected spaces X, Y is said to be **n-connected** for an integer $n \geq 0$ if the induced map*

$$f_* : [K, X] \to [K, Y] \tag{2.4}$$

is bijective for each CW complex K of dimension $< n$, and surjective for each CW complex K of dimension $\leq n$. □

An n-connected map is also said to be an n**-equivalence**. It is usual to say that a weak equivalence is ∞**-connected**. Any homotopy equivalence is n-connected for all n, as is any weak equivalence. Also, an n-equivalence is an m-equivalence for each $m \leq n$, so we can easily check the following composition properties.

Proposition 2.5 *let $f : X \to Y$ and $g : Y \to Z$ be maps.*

1) If f, g are both n-connected, then $gf : X \to Y$ is n-connected.

2) If f is $(n-1)$-connected and gf is n-connected, then g is n-connected.

3) If g is n-connected and gf is $(n-1)$-connected, then f is $(n-1)$-connected. ∎

There is a counterpart to Theorem 2.

Theorem 2.6 *A map $\alpha : X \to Y$ between path-connected spaces is an n-equivalence if and only if*

$$\alpha_* : \pi_k X \to \pi_k Y \tag{2.5}$$

is an isomorphism for each $1 \leq k < n$ and an epimorphism for each $1 \leq k \leq n$. $\blacksquare$

Corollary 2.7 *Let $\alpha : X \to Y$ be a map between connected CW complexes such that*

$$\alpha_* : \pi_k X \to \pi_k Y \tag{2.6}$$

is an isomorphism for each $1 \leq k < n$ and an epimorphism for each $1 \leq k \leq n$. Assume that $\dim X < n$ and $\dim Y \leq n$. Then α is a homotopy equivalence. $\blacksquare$

The proof of this runs the same as that in Exercise 2.1. The reader is encouraged to pay attention to the dimensional requirements we have stated.

Definition 3 *A path-connected space X is said to be n-connected if the map $* \to X$ is n-connected.* $\square$

The following are equivalent characterisations.

Proposition 2.8 *Given a path-connected space X, the following statements are equivalent*

1) *X is n-connected.*

2) *$X \to *$ is $(n+1)$-connected.*

3) *$\pi_k X = 0$ for each $1 \leq k \leq n$.* $\blacksquare$

Exercise 2.3 Let $F \to E \to B$ be a fibration sequence. Suppose that any two of the three spaces are n-connected and determine the connectivity of the third. Conclude that a map $f : X \to Y$ between path-connected spaces is n-connected if and only if its homotopy fibre F_f is $(n-1)$-connected. $\square$

Similarly we can determine the connectivity of ΩX from that of X and the connectivity of $X \times Y$ from those of X and Y.

Exercise 2.4 Suppose given a pullback diagram

$$\begin{tikzcd} \alpha^* E \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner"] & E \arrow[d, "p"] \\ A \arrow[r, "\alpha"] & B. \end{tikzcd} \tag{2.7}$$

Assume that A, B, E are path-connected and that p is a fibration. Show that p is n-connected if and only if f is n-connected. $\square$

The final exercise is included to warn against cavalier quotation of Whitehead's Theorem 2.3.

Exercise 2.5 Show that the two spaces S^2 and $S^3 \times \mathbb{C}P^\infty$ have isomorphic homotopy groups in all dimensions but are not homotopy equivalent. These spaces are CW complexes. What is out of place? $\square$